

ON EXCEPTIONAL QUOTIENT SINGULARITIES

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ABSTRACT. We study exceptional quotient singularities. In particular, we prove an exceptionality criterion in terms of the α -invariant of Tian, and utilize it to classify four-dimensional and five-dimensional exceptional quotient singularities.

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We assume that all varieties are projective, normal, and defined over \mathbb{C} .

1. INTRODUCTION

Let X be a smooth Fano variety (see [19]), let $g = g_{i\bar{j}}$ be a Kähler metric with a Kähler form

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j \in c_1(X).$$

Definition 1.1. The metric g is a Kähler–Einstein metric if

$$\mathrm{Ric}(\omega) = \omega,$$

where $\mathrm{Ric}(\omega)$ is a Ricci curvature of the metric g .

Let $\bar{G} \subset \mathrm{Aut}(X)$ be a compact subgroup. Suppose that g is \bar{G} -invariant.

Definition 1.2. Let $P_{\bar{G}}(X, g)$ be the set of C^2 -smooth \bar{G} -invariant functions φ such that

$$\omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0$$

and $\sup_X \varphi = 0$. Then the \bar{G} -invariant α -invariant of the variety X is the number

$$\alpha_{\bar{G}}(X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \exists C \in \mathbb{R} \text{ such that } \int_X e^{-\lambda\varphi} \omega^n \leq C \text{ for any } \varphi \in P_{\bar{G}}(X, g) \right\}.$$

The number $\alpha_{\bar{G}}(X)$ was introduced in [42] and [44].

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Theorem 1.3 ([42]). The Fano variety X admits a \bar{G} -invariant Kähler–Einstein metric if

$$\alpha_{\bar{G}}(X) > \frac{\dim(X)}{\dim(X) + 1}.$$

The normalized Kähler–Ricci flow on the smooth Fano X is defined by the equation

$$(1.4) \quad \begin{cases} \frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) + \omega(t), \\ \omega(0) = \omega, \end{cases}$$

where $\omega(t)$ is a Kähler form such that $\omega(t) \in c_1(X)$, and $t \in \mathbb{R}_{\geq 0}$.

Remark 1.5. It follows from [8] that the solution $\omega(t)$ to (1.4) exists for every $t > 0$.

The normalized Kähler–Ricci iteration on the smooth Fano X is defined by the equation

$$(1.6) \quad \begin{cases} \omega_{n-1} = \text{Ric}(\omega_n), \\ \omega_0 = \omega, \end{cases}$$

where ω_n is a Kähler form such that $\omega_n \in c_1(X)$.

Remark 1.7. It follows from [48] that the solution ω_n to 1.6 exists for every $n \geq 1$.

Suppose that X admits a Kähler–Einstein metric with a Kähler form ω_{KE} .

Theorem 1.8 ([45]). Any solution to (1.4) converges to ω_{KE} in the sense of Cheeger–Gromov.

Theorem 1.9 ([36]). Any solution to (1.6) converges to ω_{KE} in $C^\infty(X)$ -topology if $\alpha_{\bar{G}}(X) > 1$.

Smooth Fano manifolds that satisfy all hypotheses of Theorem 1.9 do exist.

Example 1.10. Let X be a smooth del Pezzo surface such that $K_X^2 = 5$. Then

$$\text{Aut}(X) \cong \mathbb{S}_5,$$

and $\alpha_{\bar{G}}(X) = 2$ (see [9, Example 1.11] and [10, Appendix A]), where $\bar{G} \cong \mathbb{S}_5$ or $\bar{G} \cong \mathbb{A}_5$.

Suppose, in addition, that $X \cong \mathbb{P}^n$.

Remark 1.11. If \bar{G} is the maximal compact subgroup of $\text{Aut}(\mathbb{P}^n)$, then

$$\alpha_{\bar{G}}(\mathbb{P}^n) = +\infty,$$

and the initial metric g is the Fubini–Studi metric, which is a Kähler–Einstein metric.

Yanir Rubinstein asked the following question in the Spring 2009.

Question 1.12. Is there a finite subgroup $\bar{G} \subset \text{Aut}(\mathbb{P}^n)$ such that $\alpha_{\bar{G}}(\mathbb{P}^n) > 1$?

This paper is inspired by Question 1.12.

Remark 1.13. It follows from [10, Theorem A.3] and Theorems 1.38, 1.39, 1.40, 1.41 and 3.17 that the answer to Question 1.12 is positive in the case when $n \leq 4$ (cf. Conjecture 1.18).

Let $(V \ni O)$ be a germ of Kawamata log terminal singularity (see [23, Definition 3.5]).

Definition 1.14 ([39, Definition 1.5]). The singularity $(V \ni O)$ is said to be exceptional if for every effective \mathbb{Q} -divisor D_V on the variety V one of the following two conditions is satisfied:

- either the log pair (V, D_V) is not log canonical,

- or for every resolution of singularities $\pi: U \rightarrow V$ there exists at most one π -exceptional divisor $E_1 \subset U$ such that $a(V, D_V, E_1) = -1$, where

$$K_U + D_U \sim_{\mathbb{Q}} \pi^*(K_V + D_V) + a(V, D_V, E_1)E_1 + \sum_{i=2}^r a(V, D_V, E_i)E_i,$$

the divisors E_2, \dots, E_r are π -exceptional divisors different from E_1 , and D_U is the proper transform of the divisor D_V on the variety U .

Note that the notion of exceptional Kawamata log terminal singularities is a direct generalization of the Du Val singularities of type \mathbb{E}_6 , \mathbb{E}_7 and \mathbb{E}_8 (cf. Theorem 1.38).

Remark 1.15. If V is smooth, then $(V \ni O)$ is exceptional if and only if $\dim(V) = 1$.

It follows from [38], [18] and [28] that exceptional Kawamata log terminal singularities do exist in dimensions 2 and 3. The existence in dimension 4 follows from [20] and [32, Theorem 4.9].

Theorem 1.16. Exceptional Kawamata log terminal singularities exist in every dimension.

Proof. Suppose that $(V \ni O)$ is a Brieskorn–Pham hypersurface singularity

$$\sum_{i=0}^n x_i^{a_i} = 0 \subset \mathbb{C}^{n+1} \cong \operatorname{Spec}(\mathbb{C}[x_0, x_1, \dots, x_n]),$$

where $n \geq 3$ and $2 \leq a_0 < a_1 < \dots < a_n$. Arguing as in the proof of [4, Theorem 34], we see that it follows from Theorem 3.12 that the singularity $(V \ni O)$ is exceptional if

$$1 < \sum_{i=0}^n \frac{1}{a_i} < 1 + \min\left\{\frac{1}{a_0}, \frac{1}{a_1}, \dots, \frac{1}{a_n}\right\}$$

and a_0, a_1, \dots, a_n are pairwise coprime. This is satisfied if a_0, a_1, \dots, a_n are primes and

$$(1.17) \quad \frac{1}{a_0} + \frac{1}{a_1} + \dots + \frac{1}{a_{n-1}} < 1 < \frac{1}{a_0} + \frac{1}{a_1} + \dots + \frac{1}{a_{n-1}} + \frac{1}{a_n}.$$

Let us use induction to construct the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) such that

- the numbers a_0, a_1, \dots, a_n are prime integers,
- the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) satisfies the inequalities 1.17.

If $n = 3$, then the four-tuple $(a_0, a_1, a_2, a_3) = (2, 3, 7, 41)$ satisfies the inequalities 1.17.

Suppose that $n \geq 4$, and there are primes numbers $2 \leq c_0 < c_1 < c_2 < \dots < c_{n-1}$ such that

$$\frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_{n-2}} < 1 < \frac{1}{c_0} + \frac{1}{c_1} + \dots + \frac{1}{c_{n-2}} + \frac{1}{c_{n-1}},$$

and assume that $c_{n-1} > 8$ is the largest prime with these properties (for the fixed c_0, \dots, c_{n-2}).

It follows from $c_{n-1} > 8$ that there are prime numbers p_1, p_2 and p_3 such that the inequalities

$$c_{n-1} < p_1 < p_2 < p_3 < 2c_{n-1}$$

hold (see [35, p. 209, (18)]). Put $(a_0, a_1, \dots, a_n) = (c_0, \dots, c_{n-2}, p_2, p_3)$. Then

$$\sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_2} < \sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_1} \leq 1 < \sum_{i=0}^{n-2} \frac{1}{c_i} + \frac{1}{2c_{n-1}} + \frac{1}{2c_{n-1}} < \sum_{i=0}^{n-2} \frac{1}{a_i} + \frac{1}{p_2} + \frac{1}{p_3}$$

by the maximality assumption imposed on c_{n-1} . Hence the $(n+1)$ -tuple (a_0, a_1, \dots, a_n) satisfies the inequalities 1.17, which completes the proof¹. \square

The purpose of this paper is to study exceptional quotient singularities (cf. [28], [33]).

¹ Alternatively, one can use the Sylvester sequence to construct (a_0, \dots, a_n) explicitly (suggested by S. Galkin).

Conjecture 1.18. Exceptional quotient singularities exist in every dimension.

Exceptional quotient singularities do exist in dimensions 2 and 3 (see [39], [28]).

Remark 1.19. Quotient singularities are Kawamata log terminal by [23, Proposition 3.16].

Let G be a finite subgroup in $\mathbb{GL}(n+1, \mathbb{C})$, where $n \geq 1$. Put

$$\bar{G} = \phi(G),$$

where $\phi: \mathbb{GL}(n+1, \mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^n) \cong \mathbb{PGL}(n+1, \mathbb{C})$ is the natural projection. Put

$$\text{lct}(\mathbb{P}^n, \bar{G}) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (\mathbb{P}^n, \lambda D) \text{ has log canonical singularities} \\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n} \end{array} \right. \right\} \in \mathbb{R},$$

and denote by $Z(G)$ and $[G, G]$ the center and the commutator of group G , respectively. Then

$$\text{lct}(\mathbb{P}^n, \bar{G}) \leq \frac{d}{n+1}$$

if the group G has a semi-invariant of degree d .

Remark 1.20. It follows from [10, Appendix A] that $\text{lct}(\mathbb{P}^n, \bar{G}) = \alpha_{\bar{G}}(\mathbb{P}^n)$.

Suppose that $(V \ni O)$ is a quotient singularity \mathbb{C}^{n+1}/G .

Remark 1.21. Let $R \subseteq G$ be a subgroup generated by all reflections (see [40, §4.1]). Then

- the subgroup $R \subseteq G$ is normal,
- the quotient \mathbb{C}^{n+1}/R is isomorphic to \mathbb{C}^{n+1} (see [37], [40, Theorem 4.2.5]),
- the subgroup R is trivial if $G \subset \mathbb{SL}(n+1, \mathbb{C})$.

In particular, if G is a trivial group, then $(V \ni O)$ is not exceptional by Remark 1.15.

Example 1.22. Suppose that G is the subgroup number 32 in [37, Table VII]. Then

- the group G is generated by reflections (see [37]),
- the singularity $(V \ni O)$ is not exceptional by Remark 1.21,
- it follows from Theorem 1.40 that $\text{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$.

Suppose that G does not contain reflections (this is always the case if $G \subset \mathbb{SL}(n+1, \mathbb{C})$).

Theorem 1.23. The following assertions hold:

- the singularity $(V \ni O)$ is exceptional if $\text{lct}(\mathbb{P}^n, \bar{G}) > 1$,
- the singularity $(V \ni O)$ is not exceptional if one of the following conditions are satisfied:
 - the inequality $\text{lct}(\mathbb{P}^n, \bar{G}) < 1$ hold,
 - G has a semi-invariant of degree at most $\dim(V) = n+1$,
- for any subgroup $G' \subset \mathbb{GL}(n+1, \mathbb{C})$ such that

$$\phi(G') = \bar{G},$$

the singularity $(V \ni O)$ is exceptional \iff the singularity \mathbb{C}^{n+1}/G' is exceptional.

Proof. The assertions follow from Theorem 3.16 (cf. [33, Proposition 3.1], [33, Lemma 3.1]). \square

The assumption that G contains no reflections is crucial for Theorem 1.23 (see Example 1.22).

Conjecture 1.24. The singularity $(V \ni O)$ is exceptional $\iff \text{lct}(\mathbb{P}^n, \bar{G}) > 1$.

Note that Conjecture 1.24 is a special case of Conjecture 3.6 (cf. [43, Question 1]).

Remark 1.25. A semi-invariant of the group G is its invariant if

$$Z(G) \subseteq [G, G]$$

and \bar{G} is a non-abelian simple group.

By Theorem 1.23 and [40, § 4.5], if $\dim(V) = 2$, then the following assertions are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the subgroup G has no semi-invariants of degree at most 2,
- the inequality $\text{lct}(\mathbb{P}^1, \bar{G}) \geq 3/2$ holds.

Theorem 1.26 ([28, Theorem 1.2]). If $\dim(V) = 3$, then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the subgroup G does not have semi-invariants of degree at most 3.

By Theorem 3.17, if $\dim(V) = 3$, then $(V \ni O)$ is exceptional $\iff \text{lct}(\mathbb{P}^2, \bar{G}) \geq 4/3$.

Example 1.27 ([33, Example 3.1]). Let $\Gamma \subset \mathbb{SL}(2, \mathbb{C})$ be a binary icosahedron group. Put

$$G = \left\{ \left(\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} \right) \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \Gamma \ni \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right. \right\} \subset \mathbb{SL}(4, \mathbb{C}),$$

where $a_{ij} \in \mathbb{C} \ni b_{ij}$. Then G does not have semi-invariants of degree at most 4, because Γ does not have semi-invariants of degree at most 4 (see [40, § 4.5]). It follows from [33, Proposition 2.1] that the singularity $(V \ni O)$ is not exceptional (cf. Theorem 1.29).

Hence, the assertion of Theorem 1.26 is no longer true if $\dim(V) \geq 4$.

Definition 1.28 ([3]). The subgroup $G \subset \mathbb{GL}(n+1, \mathbb{C})$ is said to be

- transitive if the corresponding $(n+1)$ -dimensional representation is irreducible,
- primitive if there is no non-trivial decomposition

$$\mathbb{C}^{n+1} = \bigoplus_{i=1}^r V_i$$

such that $g(V_i) = V_j$ for all $g \in G$.

Note that G is transitive if it is primitive. Note also that $\bar{G} \cong G/Z(G)$ if G is transitive.

Theorem 1.29 ([33, Proposition 2.1]). If $(V \ni O)$ is exceptional, then G is primitive.

Proof. Suppose that G is not primitive. Then there is a non-trivial decomposition

$$\text{Spec}(\mathbb{C}[x_0, x_1, \dots, x_n]) \cong \mathbb{C}^{n+1} = \bigoplus_{i=1}^r V_i$$

such that $g(V_i) = V_j$ for all $g \in G$. We may assume that $\dim(V_1) \leq \dots \leq \dim(V_r)$.

Put $d = \dim(V_1)$. Then $d \leq \lfloor (n+1)/2 \rfloor$. We may assume that $V_1 \subset \mathbb{C}^{n+1}$ is given by

$$x_d = x_{d+1} = x_{d+2} = \dots = x_n = 0.$$

Let \mathcal{M}_1 be a linear system on \mathbb{P}^n that consists of hyperplanes that are given by

$$\sum_{i=0}^{d-1} \lambda_i x_i = 0 \subset \mathbb{P}^n \cong \text{Proj}(\mathbb{C}[x_0, x_1, \dots, x_n]),$$

where $\lambda_i \in \mathbb{C}$. Let $\mathcal{M}_1, \dots, \mathcal{M}_s$ be the \bar{G} -orbit of the linear system \mathcal{M}_1 . Then

$$\frac{n+1}{s} \left(\sum_{i=1}^s \mathcal{M}_i \right) \sim_{\mathbb{Q}} -K_{\mathbb{P}^n},$$

where $s \leq \lfloor (n+1)/d \rfloor$. Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace that is given by $x_0 = \dots = x_d = 0$. Then

$$\frac{n+1}{s} \text{mult}_\Lambda \left(\sum_{i=1}^s \mathcal{M}_i \right) \geq \frac{n+1}{s} \text{mult}_\Lambda (\mathcal{M}_1) = \frac{n+1}{s} \geq d = n - \dim(\Lambda),$$

which implies that $(V \ni O)$ is not exceptional by Lemma 3.5 and Theorem 3.16. \square

Up to conjugation, there are finitely many primitive finite subgroups in $\mathbb{SL}(n+1, \mathbb{C})$ (see [11]).

Theorem 1.30. Suppose that $\text{lct}(\mathbb{P}^n, \bar{G}) \geq 1$. Then G is transitive.

Proof. Arguing as in the proof of Theorem 1.29, we obtain the required assertion. \square

Using [40, § 4.5], [49] and direct computation, we see that it follows from Theorem 1.29 that

$$\text{lct}(\mathbb{P}^n, \bar{G}) \leq \begin{cases} 6 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 3 & \text{if } n = 3. \end{cases}$$

Theorem 1.31. The inequality $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 4(n+1)$ holds for every $n \geq 1$.

Proof. Let p be any prime number which does not divide $|G|$. Then

$$\text{lct}(\mathbb{P}^n, \bar{G}) \leq p - 1,$$

because G has a semi-invariant of degree at most $(p-1)(n+1)$ by [41, Lemma 2].

By the Bertrand's postulate (see [35]), there is a prime number p' such that

$$2n+3 < p' < 2(2n+3),$$

which implies that $p' \leq 4n+5$. Moreover, it follows from Theorems 1.23 and 1.29 that we may assume that G is primitive. Hence p' does not divide $|G|$ by [14, Theorem 1]. \square

Arguing as in the proof of Theorem 1.31 and using [30], we obtain the following result.

Corollary 1.32. Suppose that $n \geq 23$. Then $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 12(n+1)/5$.

In fact, we expect the following to be true (cf. [41]).

Conjecture 1.33. There is $\alpha \in \mathbb{R}$ such that $\text{lct}(\mathbb{P}^n, \bar{G}) \leq \alpha$ for any $\bar{G} \subset \text{Aut}(\mathbb{P}^n)$ and $n \geq 1$.

Primitive finite subgroups of $\mathbb{SL}(n+1, \mathbb{C})$ are classified in [3], [5], [26], [46], [47] for $n \leq 6$.

Example 1.34 (see [3, §123], [31]). Let \mathbb{H} be a subgroup in $\mathbb{SL}(4, \mathbb{C})$ that is generated by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and let $N \subset \mathbb{SL}(4, \mathbb{C})$ be the normalizer of the subgroup \mathbb{H} . Then there is an exact sequence

$$1 \longrightarrow \tilde{\mathbb{H}} \xrightarrow{\alpha} N \xrightarrow{\beta} \mathbb{S}_6 \longrightarrow 1,$$

where $\tilde{\mathbb{H}} = \langle \mathbb{H}, \text{diag}(\sqrt{-1}) \rangle$. The following four subgroups of the group $\mathbb{SL}(4, \mathbb{C})$ are primitive:

- the group $G_1 = N \subset \mathbb{SL}(4, \mathbb{C})$,
- a subgroup $G_2 \subsetneq N$ such that $\alpha(\tilde{\mathbb{H}}) \subsetneq G_2$ and $\beta(G_2) \cong \mathbb{A}_6$,
- a subgroup $G_3 \subsetneq N$ such that $\alpha(\tilde{\mathbb{H}}) \subsetneq G_3$ and $\beta(G_3) \cong \mathbb{S}_5$, where the embedding

$$\beta(G_3) \subset \mathbb{S}_6$$

is non-standard, i. e. the standard one twisted by an outer automorphism of \mathbb{S}_6 ,

- a subgroup $G_4 \subset G_3$ such that $\alpha(\tilde{\mathbb{H}}) \subsetneq G_4$ and $\beta(G_4) \cong \mathbb{A}_5$.

Example 1.35 (cf. Appendix A). Let \mathbb{H} be the Heisenberg group of all unipotent 3×3 -matrices with entries in \mathbb{F}_5 . Then there are a monomorphism $\rho: \mathbb{H} \rightarrow \mathrm{SL}(5, \mathbb{C})$ and an exact sequence

$$1 \longrightarrow \mathbb{H} \xrightarrow{\alpha} \mathrm{HIM} \xrightarrow{\beta} \mathrm{SL}(2, \mathbb{F}_5) \longrightarrow 1,$$

where $\mathrm{HIM} \subset \mathrm{SL}(5, \mathbb{C})$ is the normalizer of the subgroup $\rho(\mathbb{H})$. Moreover,

- the subgroup $\mathrm{HIM} \subset \mathrm{SL}(5, \mathbb{C})$ is primitive (see [5, Theorem 9A], [16]),
- the subgroup of HIM of index 5 that contains $\alpha(\mathbb{H})$ is primitive.

The main purpose of this paper is to prove the following result (cf. Theorem 1.26).

Theorem 1.36. Suppose that $\dim(V) \leq 5$. Then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the group G is primitive and has no semi-invariants of degree at most $\dim(V)$.

Proof. The required assertion follows from Theorems 1.23, 1.26, 1.29, 1.40 and 1.41. \square

The assertion of Theorem 1.36 is no longer true if $\dim(V) \geq 6$ (see Example 3.20).

Remark 1.37. There exists a finite subgroup $G' \subset \mathrm{SL}(n+1, \mathbb{C})$ such that

$$\phi(G') = \bar{G} \subset \mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1, \mathbb{C}),$$

and $(V \ni O)$ is exceptional \iff the singularity \mathbb{C}^{n+1}/G' is exceptional (see Theorem 1.23).

Suppose, in addition, that $G \subset \mathrm{SL}(n+1, \mathbb{C})$.

Theorem 1.38 ([38, Example 5.2.3]). If $\dim(V) = 2$, then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the group \bar{G} is isomorphic to one of the groups \mathbb{A}_4 , \mathbb{S}_4 or \mathbb{A}_5 .

Theorem 1.39 ([28, Theorem 3.13]). If $\dim(V) = 3$, then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- either $\bar{G} \cong \mathrm{PSL}(2, \mathbb{F}_7)$, or $\bar{G} \cong \mathbb{A}_6$, or G is isomorphic to one of the following groups:
 - the Hessian group, which can be characterized by the exact sequence

$$1 \longrightarrow \mathbb{H}(3, \mathbb{F}_3) \longrightarrow G \longrightarrow \mathbb{S}_4 \longrightarrow 1,$$

where $\mathbb{H}(3, \mathbb{F}_3)$ is the Heisenberg group of unipotent 3×3 -matrices with entries in \mathbb{F}_3 ,
 – the normal subgroup of the Hessian group of index 3 that contains $\mathbb{H}_3(\mathbb{F}_3)$.

In this paper we prove the following two results (cf. Remark 1.37).

Theorem 1.40. Suppose that $\dim(V) = 4$. Then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the inequality $\mathrm{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$ holds,
- either $\bar{G} \cong \mathbb{A}_6$, or $\bar{G} \cong \mathbb{S}_6$, or $\bar{G} \cong \mathbb{A}_7$, or

$$\bar{G} \cong [\mathbb{O}(5, \mathbb{F}_3), \mathbb{O}(5, \mathbb{F}_3)],$$

or G is isomorphic to one of the four groups constructed in Example 1.34.

Proof. The required assertion follows from Theorems 1.23, 4.3 and 4.6 and Lemma 4.9. \square

Theorem 1.41. Suppose that $\dim(V) = 5$. Then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the inequality $\mathrm{lct}(\mathbb{P}^4, \bar{G}) \geq 6/5$ holds,

- the group G is isomorphic to one of the two groups constructed in Example 1.35.

Proof. The required assertion follows from Theorems 1.23, 5.5, 5.1 and Lemmas 5.3 and 5.2. \square

The structure of the paper is the following:

- in Section 2 we collect auxiliary results,
- in Section 3 we prove the exceptionality criterion for $(V \ni O)$,
- in Sections 4 we prove results that are used in the proof of Theorem 1.40,
- in Sections 5 we prove results that are used in the proof of Theorem 1.41,
- in Appendix A we prove Corollary A.2 and Theorem A.9 that are used in Section 5.

Many of our results can be obtained by direct computations using information from [12].

Throughout the paper we use the following standard notation:

- the symbol \mathbb{Z}_n denotes the cyclic group of order n ,
- the symbol \mathbb{F}_n denotes the finite field consisting of n elements,
- the symbol \mathbb{S}_n denotes the symmetric group of degree n ,
- the symbol \mathbb{A}_n denotes the alternating group of degree n ,
- the symbols $\mathrm{SL}(m, \mathbb{F}_n)$, $\mathrm{PSL}(m, \mathbb{F}_n)$, $\mathbb{O}(m, \mathbb{F}_n)$ denote the corresponding linear groups,
- the symbol $k.G$ denotes any central extension of a group G with the center \mathbb{Z}_k .

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2. PRELIMINARIES

Let X be a variety, let B_X and D_X be effective \mathbb{Q} -divisors on the variety X such that the singularities of the log pair (X, B_X) are Kawamata log terminal (see [23, Definition 3.5]), and

$$K_X + B_X + D_X$$

is a \mathbb{Q} -Cartier divisor. Let $Z \subseteq X$ be a closed non-empty subvariety.

Definition 2.1. The log canonical threshold of the boundary D_X along Z is

$$c_Z(X, B_X, D_X) = \sup \left\{ \lambda \in \mathbb{Q} \mid \text{the pair } (X, B_X + \lambda D_X) \text{ is log canonical along } Z \right\}.$$

For simplicity, we put $c(X, B_X, D_X) = c_X(X, B_X, D_X)$. We put

$$c_Z(X, D_X) = c_Z(X, B_X, D_X)$$

in the case when $B_X = 0$. For simplicity, we also put $c(X, D_X) = c_X(X, D_X)$.

Remark 2.2. The following conditions are equivalent:

- the log pair $(X, B_X + D_X)$ is Kawamata log terminal along Z ,
- the inequality $c_Z(X, B_X, D_X) > 1$ holds.

Let $\mathrm{LCS}(X, B_X + D_X) \subset X$ be the subset such that

$$P \in \mathrm{LCS}(X, B_X + D_X) \iff c_P(X, B_X, D_X) \leq 1$$

for every point $P \in X$. Then the subset $\mathrm{LCS}(X, B_X + D_X) \subset X$ is called the locus of log canonical singularities of the log pair $(X, B_X + D_X)$ (see [38]).

Theorem 2.3 ([38, Lemma 5.7]). Suppose that $-(K_X + B_X + D_X)$ is nef and big. Then

$$\mathrm{LCS}(X, B_X + D_X)$$

is connected.

Let $\pi: \bar{X} \rightarrow X$ be a birational morphism such that \bar{X} is smooth. Then

$$K_{\bar{X}} + B_{\bar{X}} + D_{\bar{X}} \sim_{\mathbb{Q}} \pi^* (K_X + B_X + D_X) + \sum_{i=1}^m d_i E_i,$$

where $B_{\bar{X}}$ and $D_{\bar{X}}$ are proper transforms of the divisors B_X and D_X on the variety \bar{X} , respectively, $d_i \in \mathbb{Q}$, and E_i is an exceptional divisor of the morphism π . Note that

$$B_{\bar{X}} + D_{\bar{X}} = \sum_{i=1}^r a_i G_i,$$

where $a_i \in \mathbb{Q}_{\geq 0}$, and G_i is prime Weil divisor on \bar{X} . We may assume that

$$\left(\bigcup_{i=1}^r G_i \right) \cup \left(\bigcup_{i=1}^m E_i \right)$$

is a divisor with simple normal crossing. Put

$$\mathcal{I}(X, B_X + D_X) = \pi_* \left(\sum_{i=1}^m [d_i] E_i - \sum_{i=1}^r [a_i] G_i \right).$$

Remark 2.4. The sheaf $\mathcal{I}(X, B_X + D_X)$ is an ideal sheaf, because $a_i \geq 0$ for every $i \in \{1, \dots, r\}$.

Let $\mathcal{L}(X, B_X + D_X)$ be a subscheme that corresponds to $\mathcal{I}(X, B_X + D_X)$. Then

$$\text{Supp} \left(\mathcal{L}(X, B_X + D_X) \right) = \text{LCS}(X, B_X + D_X) \subset X,$$

and $\mathcal{I}(X, B_X)$ is known as the multiplier ideal sheaf of the log pair (X, B_X) (see [25]).

Remark 2.5. If $(X, B_X + D_X)$ is log canonical, then $\mathcal{L}(X, B_X + D_X)$ is reduced.

Note that the subscheme $\mathcal{L}(X, B_X + D_X)$ is known as the subscheme of log canonical singularities of the log pair $(X, B_X + D_X)$. The subscheme $\mathcal{L}(X, B_X + D_X)$ was introduced in [38].

Theorem 2.6 ([25, Theorem 9.4.8]). Let H be a nef and big \mathbb{Q} -divisor on X such that

$$K_X + B_X + D_X + H \equiv D$$

for some Cartier divisor D on the variety X . Take $i \geq 1$. Then

$$H^i \left(\mathcal{I}(X, B_X + D_X) \otimes D \right) = 0.$$

Let $\text{LCS}(X, B_X + D_X)$ be the set that consists of all possible centers of log canonical singularities of the log pair $(X, B_X + D_X)$ (see [10, Definition 2.2]).

Remark 2.7. Let \mathcal{H} be a linear system on the variety X that has no base points. Put

$$Z \cap H = \sum_{i=1}^k Z_i,$$

where H is a general divisor in \mathcal{H} , and $Z_i \subset H$ is an irreducible subvariety. Then

$$Z \in \text{LCS}(X, B_X + D_X) \iff \{Z_1, \dots, Z_k\} \subseteq \text{LCS} \left(H, (B_X + D_X)|_H \right).$$

Suppose that $Z \in \text{LCS}(X, B_X + D_X)$.

Definition 2.8. The subvariety Z is a minimal center in $\mathbb{LCS}(X, B_X + D_X)$ if the set

$$\mathbb{LCS}(X, B_X + D_X)$$

does not contain subvarieties of the variety X that are proper subsets of the subvariety Z .

Suppose that $(X, B_X + D_X)$ is log canonical. Then $\mathcal{L}(X, B_X + D_X)$ is reduced by Remark 2.5.

Lemma 2.9 ([21, Proposition 1.5]). Let Z' be a center in $\mathbb{LCS}(X, B_X + D_X)$ such that

$$\emptyset \neq Z \cap Z' = \sum_{i=1}^k Z_i,$$

where $Z_i \subsetneq Z$ is an irreducible subvariety. Then $Z_i \in \mathbb{LCS}(X, B_X + D_X)$ for every $i \in \{1, \dots, k\}$.

Suppose that Z is a minimal center in $\mathbb{LCS}(X, B_X + D_X)$.

Theorem 2.10 ([22, Theorem 1]). Let Δ be an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Then

- the variety Z is normal and has at most rational singularities,
- there exists an effective \mathbb{Q} -divisor B_Z on the variety Z such that

$$\left(K_X + B_X + D_X + \Delta \right) \Big|_Z \sim_{\mathbb{Q}} K_Z + B_Z,$$

and (Z, B_Z) has Kawamata log terminal singularities.

Let $\bar{G} \subseteq \text{Aut}(X)$ be a finite subgroup such that B_X and D_X are \bar{G} -invariant. Then

$$g(Z) \in \mathbb{LCS}(X, B_X + D_X)$$

for every $g \in \bar{G}$, and the locus $\mathbb{LCS}(X, B_X + D_X)$ is \bar{G} -invariant. But

$$g(Z) \cap g'(Z) \neq \emptyset \iff g(Z) = g'(Z)$$

for every $g \in \bar{G} \ni g'$, because Z is a minimal center in $\mathbb{LCS}(X, B_X + D_X)$.

Lemma 2.11. Suppose that the divisor $B_X + D_X$ is ample. Let ϵ be an arbitrary rational number such that $\epsilon > 1$. Then there exists an effective \bar{G} -invariant \mathbb{Q} -divisor D on the variety X such that

$$\mathbb{LCS}(X, D) = \bigcup_{g \in \bar{G}} \{g(Z)\},$$

the log pair (X, D) is log canonical, and the equivalence $D \sim_{\mathbb{Q}} \epsilon(B_X + D_X)$ holds.

Proof. Take $m \in \mathbb{Z}$ such that $m(B_X + D_X)$ is a very ample Cartier divisor. Take a general divisor

$$R \in \left| nm(B_X + D_X) \right|$$

such that $Z \subset \text{Supp}(R)$ and R is \bar{G} -invariant, where $n \gg 0$. Then

$$\bigcup_{g \in \bar{G}} \{g(Z)\} \subseteq \mathbb{LCS}\left(X, \lambda(B_X + D_X) + \mu R\right) \subseteq \mathbb{LCS}(X, B_X + D_X)$$

for some positive rational numbers λ and μ such that $\lambda < 1 \leq \lambda + \mu nm < \epsilon$. One has

$$\lambda(B_X + D_X) + \mu R \sim_{\mathbb{Q}} (\lambda + \mu nm)(B_X + D_X).$$

It follows from the generality of the divisor R that $(X, \mu R)$ is log terminal, and

$$\mathbb{LCS}\left(X, \lambda(B_X + D_X) + \mu R\right) = \bigcup_{g \in \bar{G}} g(Z),$$

because $\lambda < 1$ and $n \gg 0$. Then there is $\theta \in \mathbb{Q}_{>0}$ such that $0 < 1 - \theta\mu \leq \lambda < 1$ and

$$\bigcup_{g \in \bar{G}} \{g(Z)\} \subseteq \text{LCS}\left(X, (1 - \theta\mu)(B_X + D_X) + \mu R\right) \subseteq \text{LCS}\left(X, \lambda(B_X + D_X) + \mu R\right),$$

but the log pair $(X, (1 - \theta\mu)(B_X + D_X) + \mu R)$ is log canonical at the general point of Z .

Note that for a fixed R , the number θ is a function of μ . In the above process, we can choose the number μ so that $1 \leq 1 - \theta\mu + \mu nm < \epsilon$ and

$$\text{LCS}\left(X, (1 - \theta\mu)(B_X + D_X) + \mu R\right) = \bigcup_{g \in \bar{G}} \{g(Z)\},$$

because Z is a minimal center in $\text{LCS}(X, B_X + D_X)$. Put

$$D = (1 - \theta\mu)(B_X + D_X) + \mu R + \frac{\epsilon - 1 - \theta\mu + \mu nm}{nm}M,$$

where M is a general \bar{G} -invariant divisor in $|R|$. Then D is the required divisor. \square

Suppose that $X \cong \mathbb{P}^n$. Let H be a hyperplane in \mathbb{P}^n .

Lemma 2.12. Suppose that $\text{LCS}(X, B_X + D_X)$ is an equidimensional subvariety in \mathbb{P}^n . Put

$$r = \begin{cases} \lceil \mu - s - 1 \rceil & \text{if } \mu \in \mathbb{Z}, \\ \lceil \mu - s - 1 \rceil + 1 & \text{if } \mu \notin \mathbb{Z}, \end{cases}$$

where $s = n - \dim(\text{LCS}(X, B_X + D_X))$, and $\mu \in \mathbb{Q}$ such that $B_X + D_X \equiv \mu H$. Then $r \geq 0$ and

$$\deg(\text{LCS}(X, B_X + D_X)) \leq \binom{s+r}{r}.$$

Proof. Put $Y = \text{LCS}(X, B_X + D_X)$. Let $\Pi \subset \mathbb{P}^n$ be a general linear subspace of dimension s . Put

$$D = (B_X + D_X)|_{\Pi}$$

and $\Lambda = H \cap \Pi$. Then $\deg(Y) = |Y \cap \Pi|$ and $\text{LCS}(\Pi, D) = H \cap \Pi$ by Remark 2.7. One has

$$K_{\Pi} + D \equiv (\mu - s - 1)\Lambda.$$

It follows from Theorem 2.6 that there is an exact sequence of cohomology groups

$$0 \longrightarrow H^0\left(\mathcal{O}_{\Pi}(r\Lambda) \otimes \mathcal{I}(\Pi, D)\right) \longrightarrow H^0\left(\mathcal{O}_{\Pi}(r\Lambda)\right) \longrightarrow H^0\left(\mathcal{O}_{\mathcal{L}(\Pi, D)}\right) \longrightarrow 0,$$

and $\text{Supp}(\mathcal{L}(\Pi, D)) = \text{LCS}(\Pi, D) = Y \cap \Pi \neq \emptyset$. Therefore, we see that $r \geq 0$ and

$$\deg(Y) = |Y \cap \Pi| \leq h^0\left(\mathcal{O}_{\mathcal{L}(\Pi, D)}\right) \leq h^0\left(\mathcal{O}_{\Pi}(r\Lambda)\right) = h^0\left(\mathcal{O}_{\mathbb{P}^s}(r)\right) = \binom{s+r}{r},$$

which completes the proof. \square

Let G be a finite subgroup in $\text{GL}(n+1, \mathbb{C})$ such that

$$\bar{G} = \phi(G),$$

where $\phi: \text{GL}(n+1, \mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^n) \cong \text{PGL}(n+1, \mathbb{C})$ is the natural projection.

Lemma 2.13. Suppose that G is conjugate to a subgroup in $\text{GL}(n+1, \mathbb{R})$. Then

- the subgroup G has an invariant of degree 2,
- the inequality $\text{lct}(\mathbb{P}^n, \bar{G}) \leq 2/(n+1)$ holds.

Proof. This is obvious. \square

Suppose that Z is \bar{G} -invariant. Then there is a homomorphism $\xi: \bar{G} \rightarrow \text{Aut}(Z)$.

Lemma 2.14. If Z is not contained in a linear subspace of \mathbb{P}^n , then ξ is a monomorphism.

Proof. The required assertion immediately follows from the fact that eigenvectors that correspond to a fixed eigenvalue of any matrix in $\text{GL}(n+1, \mathbb{C})$ form a vector subspace in \mathbb{C}^{n+1} . \square

Let us conclude this section by the following well-known result.

Theorem 2.15. Let C be a smooth irreducible curve of genus $g \geq 2$. Then

$$|\text{Aut}(C)| \leq 84(g-1).$$

Proof. The required inequality is the famous Hurwitz bound (see [6, Theorem 3.17]). \square

3. EXCEPTIONALITY CRITERION

Let X be a variety, let B_X be an effective \mathbb{Q} -divisor on X such that the log pair (X, B_X) has at most Kawamata log terminal singularities, and the divisor $-(K_X + B_X)$ is ample.

Remark 3.1. The log pair (X, B_X) is called the log Fano variety.

Let $\bar{G} \subset \text{Aut}(X)$ be a finite subgroup such that the divisor B_X is \bar{G} -invariant.

Definition 3.2. Global \bar{G} -invariant log canonical threshold of the log Fano variety (X, B_X) is

$$\text{lct}(X, B_X, \bar{G}) = \inf \left\{ c(X, B_X, D_X) \in \mathbb{Q} \left| \begin{array}{l} D_X \text{ is a } \bar{G}\text{-invariant } \mathbb{Q}\text{-Cartier effective } \mathbb{Q}\text{-divisor} \\ \text{on the variety } X \text{ such that } D_X \sim_{\mathbb{Q}} -(K_X + B_X) \end{array} \right. \right\}.$$

For simplicity, we use the following notation:

- we put $\text{lct}(X, B_X, \bar{G}) = \text{lct}(X, \bar{G})$ if $B_X = 0$,
- we put $\text{lct}(X, B_X, \bar{G}) = \text{lct}(X, B_X)$ if \bar{G} is trivial,
- we put $\text{lct}(X, B_X, \bar{G}) = \text{lct}(X)$ if $B_X = 0$ and \bar{G} is trivial.

Remark 3.3. Suppose that $B_X = 0$. Put $V = X/\bar{G}$. Let $\theta: X \rightarrow V$ be the quotient map. Then

$$K_X \sim_{\mathbb{Q}} \theta^*(K_V + R_V),$$

where R_V is a ramification (\mathbb{Q} -)divisor of the morphism θ . Note that

- the divisor $-(K_V + R_V)$ is an ample \mathbb{Q} -Cartier divisor,
- the log pair (V, R_V) is Kawamata log terminal by [23, Proposition 3.16],
- it follows from [23, Proposition 3.16] that

$$\text{lct}(X, \bar{G}) = \text{lct}(V, R_V).$$

If X is smooth and $B_X = 0$, then $\text{lct}(X, \bar{G}) = \alpha_{\bar{G}}(X)$ by [10, Appendix A].

Example 3.4. Suppose that $X \cong \mathbb{P}^1$. Then

$$B_X = \sum_{i=1}^n a_i P_i,$$

where P_i is a point, and $a_i \in \mathbb{Q}$ such that $0 \leq a_i < 1$. We may assume that $a_0 \leq \dots \leq a_n$. Then

$$\text{lct}(X, B_X) = \frac{1 - a_n}{2 - \sum_{i=1}^n a_i},$$

where $\sum_{i=1}^n a_i < 2$, because the divisor $-(K_X + B_X)$ is ample. It follows from Remark 3.3 that

$$\text{lct}(X, \bar{G}) = \frac{2}{|\Lambda|},$$

where Λ is a \bar{G} -orbit of the smallest length (cf. Theorem 1.38).

Lemma 3.5. The global log canonical threshold $\text{lct}(X, B_X, \bar{G})$ is equal to

$$\inf \left\{ c \left(X, B_X, \sum_{i=1}^r a_i \mathcal{D}_i \right) \in \mathbb{Q} \left| \begin{array}{l} \mathcal{D}_i \text{ is a linear system, } a_i \in \mathbb{Q} \text{ such that } a_i \geq 0, \\ \text{the log pair } \left(X, B_X, \sum_{i=1}^r a_i \mathcal{D}_i \right) \text{ is } \bar{G}\text{-invariant,} \\ \text{the equivalence } \sum_{i=1}^r a_i \mathcal{D}_i \sim_{\mathbb{Q}} -K_X - B_X \text{ holds.} \end{array} \right. \right\}.$$

Proof. The required assertion follows from Definition 3.2 and [23, Theorem 4.8]. \square

It is unknown whether $\text{lct}(X, B_X, \bar{G}) \in \mathbb{Q}$ or not (cf. [43, Question 1]).

Conjecture 3.6. There is an effective \bar{G} -invariant \mathbb{Q} -divisor D_X on the variety X such that

$$\text{lct}(X, B_X, \bar{G}) = c(X, B_X, D_X) \in \mathbb{Q},$$

and the equivalence $D_X \sim_{\mathbb{Q}} -(K_X + B_X)$ holds.

Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and let $\pi: W \rightarrow V$ be a birational morphism such that the following hypotheses are satisfied:

- the exceptional locus of π consists of one irreducible divisor $E \subset W$ such that $O \in \pi(E)$,
- the log pair (W, E) has purely log terminal singularities (see [23, Definition 3.5]),
- the divisor $-E$ is a π -ample \mathbb{Q} -Cartier divisor.

Theorem 3.7. The birational morphism $\pi: W \rightarrow V$ does exist.

Proof. The required assertion follows from [32, Proposition 2.9], [24, Theorem 1.5] and [2]. \square

We say that $\pi: W \rightarrow V$ is a plt blow up of the singularity $(V \ni O)$.

Definition 3.8. We say that $(V \ni O)$ is weakly-exceptional if it has unique plt blow up.

Weakly-exceptional Kawamata log terminal singularities do exist (see [24, Example 2.2]).

Lemma 3.9 ([24, Corollary 1.7]). If $(V \ni O)$ is weakly-exceptional, then $\pi(E) = O$.

Suppose that $\pi(E) = O$. Let R_1, \dots, R_s be irreducible components of $\text{Sing}(W)$ such that

$$\dim(R_i) = \dim(W) - 2$$

and $R_i \subset E$ for every $i \in \{1, \dots, s\}$. Put

$$\text{Diff}_E(0) = \sum_{i=1}^s \frac{m_i - 1}{m_i} R_i,$$

where m_i is the smallest positive integer such that $m_i E$ is Cartier at a general point of R_i .

Lemma 3.10 ([23, Theorem 7.5]). The following assertions hold:

- the variety E is normal,
- the log pair $(E, \text{Diff}_E(0))$ is Kawamata log terminal.

Thus, the log pair $(E, \text{Diff}_E(0))$ is a log Fano variety, because $-E$ is π -ample.

Theorem 3.11 ([24, Theorem 2.1]). The singularity $(V \ni O)$ is weakly-exceptional if and only if

$$\text{lct}(E, \text{Diff}_E(0)) \geq 1.$$

Theorem 3.12 ([32, Theorem 4.9]). The singularity $(V \ni O)$ is exceptional if and only if

$$c(E, \text{Diff}_E(0), D_E) > 1$$

for every effective \mathbb{Q} -divisor D_E on the variety E such that $D_E \sim_{\mathbb{Q}} -(K_E + \text{Diff}_E(0))$.

Thus, if the assertion of Conjecture 3.6 is true, then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the inequality $\text{lct}(E, \text{Diff}_E(0)) > 1$ holds.

Corollary 3.13. If $(V \ni O)$ is exceptional, then $(V \ni O)$ is weakly-exceptional.

Suppose, in addition, that $(V \ni O)$ is a quotient singularity \mathbb{C}^{n+1}/G , where G is a finite subgroup in $\text{GL}(n+1, \mathbb{C})$. Suppose that G does not contain reflections (see Remark 1.21). Put

$$\bar{G} = \phi(G),$$

where $\phi: \text{GL}(n+1, \mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^n) \cong \text{PGL}(n+1, \mathbb{C})$ is the natural projection.

Remark 3.14. Let $\eta: \mathbb{C}^{n+1} \rightarrow V$ be the quotient map. Then there is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\omega} & W \\ \gamma \downarrow & & \downarrow \pi \\ \mathbb{C}^{n+1} & \xrightarrow{\eta} & V, \end{array}$$

where γ is the blow up of O , the morphism ω is the quotient map that is induced by the lifted action of G on the variety U , and π is a birational morphism. Then π is a plt blow up of $(V \ni O)$.

Thus, to prove the existence of a plt blow up of $(V \ni O)$ we do not need to use [2].

Theorem 3.15. The singularity $(V \ni O)$ is weakly-exceptional $\iff \text{lct}(\mathbb{P}^n, \bar{G}) \geq 1$.

Proof. Let us use the notation and assumptions of Remark 3.14. Put

$$E = \omega(F),$$

where $F \cong \mathbb{P}^n$ is the exceptional divisor of the morphism γ . Then $E \cong \mathbb{P}^n/\bar{G}$.

Since the group G does not contain reflections, it follows from Remark 3.3 that

$$\text{lct}(\mathbb{P}^n, \bar{G}) = \text{lct}(E, \text{Diff}_E(0)),$$

which implies that $(V \ni O)$ is weakly-exceptional if and only if $\text{lct}(\mathbb{P}^n, \bar{G}) \geq 1$ by Theorem 3.12. \square

Theorem 3.16. The following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- for any \bar{G} -invariant effective \mathbb{Q} -divisor D on \mathbb{P}^n such that

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^n},$$

the log pair (\mathbb{P}^n, D) is Kawamata log terminal.

Proof. Use the proof of Theorem 3.15 together with Theorem 3.12 and [23, Proposition 3.16]. \square

Let us show how to apply Theorems 3.15 and 3.16 (cf. [9, Example 1.9]).

Theorem 3.17. Suppose that $\dim(V) = 3$. Then the following are equivalent:

- the singularity $(V \ni O)$ is exceptional,
- the subgroup G does not have semi-invariants of degree at most 3,
- the inequality $\text{lct}(\mathbb{P}^2, \bar{G}) \geq 4/3$ holds.

Proof. Suppose that $(V \ni O)$ is exceptional. To complete the proof we must show that

$$\text{lct}(\mathbb{P}^2, \bar{G}) \geq \frac{4}{3},$$

because all remaining implications follow from Theorem 1.23.

Suppose that the strict inequality $\text{lct}(\mathbb{P}^2, \bar{G}) < 4/3$ holds. Then there exist a positive rational number $\lambda < 4/3$ and an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^2 such that

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^2} \sim \mathcal{O}_{\mathbb{P}^2}(3),$$

and $(\mathbb{P}^2, \lambda D)$ is not Kawamata log terminal. We may assume that $(\mathbb{P}^2, \lambda D)$ is log canonical.

Suppose that there is a \bar{G} -invariant curve $C \subset \mathbb{P}^2$ such that

$$\lambda D = \mu C + \Omega,$$

where μ is a rational number such that $\mu \geq 1$, and Ω is an effective \mathbb{Q} -divisor, whose support does not contain any component of the curve C . Let $L \subset \mathbb{P}^2$ be a line. Then

$$3\lambda L \sim_{\mathbb{Q}} \lambda D \sim_{\mathbb{Q}} \mu C + \Omega \sim_{\mathbb{Q}} \mu \deg(C) L + \Omega,$$

which implies that $\deg(C) < 4$. But G has no semi-invariants of degree at most 3.

Therefore, the locus $\text{LCS}(\mathbb{P}^2, \lambda D)$ is a finite \bar{G} -invariant set. Then

$$|\text{LCS}(\mathbb{P}^2, \lambda D)| \geq 4,$$

because G does not have semi-invariants of degree at most 3.

Let \mathcal{I} be the multiplier ideal sheaf of the log pair $(\mathbb{P}^2, \lambda D)$, let \mathcal{L} be the log canonical singularities subscheme of the log pair $(\mathbb{P}^2, \lambda D)$. Then there is an exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}}) \longrightarrow 0$$

by Theorem 2.6. But $h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$ and $h^0(\mathcal{O}_{\mathcal{L}}) = |\text{LCS}(\mathbb{P}^2, \lambda D)| \geq 4$. \square

Theorem 3.18. Suppose that $\dim(V) = 3$. Then the following are equivalent:

- the inequality $\text{lct}(\mathbb{P}^2, \bar{G}) \geq 1$ holds,
- the group G does not have semi-invariants of degree at most 2.

Proof. See the proof of Theorem 3.17 and use Theorem 3.15. \square

Let $G_1 \subset \text{GL}(2, \mathbb{C})$ and $G_2 \subset \text{GL}(l, \mathbb{C})$ be finite subgroups, let \mathbb{M} be the set of $2 \times l$ -matrices with entries in \mathbb{C} . For every $(g_1, g_2) \in G_1 \times G_2$ and every $M \in \mathbb{M}$, put

$$(g_1, g_2)(M) = g_1 M g_2^{-1} \in \mathbb{M} \cong \mathbb{C}^{2l},$$

which induces a homomorphism $\varphi: G_1 \times G_2 \rightarrow \text{GL}(2l, \mathbb{C})$. Note that $|\ker(\varphi)| \leq 2$ if n is even, and φ is a monomorphism if n is odd. Let $n = 2l - 1$.

Lemma 3.19. Suppose that $G = \varphi(G_1 \times G_2)$. Then $\text{lct}(\mathbb{P}^n, \bar{G}) < 1$.

Proof. Put $s = l - 1$. Let $\psi: \mathbb{P}^1 \times \mathbb{P}^s \rightarrow \mathbb{P}^n$ be the Segre embedding. Put

$$Y = \psi(\mathbb{P}^1 \times \mathbb{P}^s),$$

and let \mathcal{Q} be the linear system consisting of all quadric hypersurfaces in \mathbb{P}^n that pass through the subvariety Y . Then \mathcal{Q} is a non-empty \bar{G} -invariant linear system. The log pair

$$(\mathbb{P}^n, l\mathcal{Q})$$

is not log-canonical along Y , which implies that $\text{lct}(\mathbb{P}^n, \bar{G}) < 1$ by Lemma 3.5. \square

Let us show how to apply Lemma 3.19 (cf. Theorem 1.36).

Example 3.20. Suppose that $G = \varphi(G_1 \times G_2)$ and $l = 3$. Then

- the singularity $(V \ni O)$ is not exceptional by Theorem 1.23 and Lemma 3.19,
- the group G has no semi-invariants of degree at most 6 if $G_1 \cong 2.A_5$ or $G_2 \cong 3.A_6$.

Suppose that $l = 2$. The transposition of matrices in \mathbb{M} induces an involution $\iota \in \mathrm{GL}(4, \mathbb{C})$.

Lemma 3.21. Suppose that G is generated by $\varphi(G_1 \times G_2)$ and ι . Then $\mathrm{lct}(\mathbb{P}^3, \bar{G}) < 1$.

Proof. See the proof of Lemma 3.19. \square

4. FOUR-DIMENSIONAL CASE

Let \bar{G} be a finite subgroup in $\mathrm{Aut}(\mathbb{P}^3)$. Then there is a finite subgroup in $\mathrm{GL}(4, \mathbb{C})$ such that

$$\phi(G) = \bar{G} \subset \mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}(4, \mathbb{C})$$

where $\phi: \mathrm{GL}(4, \mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^3)$ is the natural projection.

Theorem 4.1. The inequality $\mathrm{lct}(\mathbb{P}^3, \bar{G}) \geq 1$ holds \iff the following conditions are satisfied:

- the group G is transitive,
- the group G does not have semi-invariants of degree at most 3,
- there is no \bar{G} -invariant smooth rational cubic curve in \mathbb{P}^3 .

Proof. Let us prove the \Rightarrow -part. If G has a semi-invariant of degree at most 3, then

$$\mathrm{lct}(\mathbb{P}^3, \bar{G}) \leq \frac{3}{4}$$

by Definition 3.2. If G is not transitive, then $\mathrm{lct}(\mathbb{P}^3, \bar{G}) < 1$ by Lemma 1.30.

Suppose that there is a \bar{G} -invariant smooth rational cubic curve $C \subset \mathbb{P}^3$. Let $R \subset \mathbb{P}^3$ be the surface that is swept out by lines that are tangent to C . Then

$$c(\mathbb{P}^3, R) = \frac{5}{6},$$

the surface R is \bar{G} -invariant, and $\deg(R) = 4$. Hence, we see that $\mathrm{lct}(\mathbb{P}^3, \bar{G}) < 5/6$.

Let us prove the \Leftarrow -part. Suppose that G is transitive, the subgroup G has no semi-invariants of degree at most 3, there is no \bar{G} -invariant smooth rational cubic curve in \mathbb{P}^3 , but $\mathrm{lct}(\mathbb{P}^3, \bar{G}) < 1$.

There are an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^3 such that

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^3} \sim \mathcal{O}_{\mathbb{P}^3}(4)$$

and a positive rational number $\lambda < 1$ such that $(\mathbb{P}^3, \lambda D)$ is strictly log canonical.

Let $S \subset X$ be a minimal center in $\mathrm{LCS}(\mathbb{P}^3, \lambda D)$. By Lemma 2.11, we may assume that

$$\mathrm{LCS}(\mathbb{P}^3, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\},$$

where $\dim(S) \neq 2$, because G has no semi-invariants of degree at most 3.

The locus $\mathrm{LCS}(\mathbb{P}^3, \lambda D)$ is connected by Theorem 2.3. Then S is \bar{G} -invariant by Lemma 2.9.

The group G is transitive, which implies that $\dim(S) \neq 0$.

We see that S is a curve. Then $\deg(S) \leq 3$ by Lemma 2.12, and S is not contained in a plane, because G is transitive. Hence S is a smooth rational cubic curve. \square

Combining Lemma 2.14, Theorem 4.1 and the classification of finite subgroups in $\mathrm{PSL}(2, \mathbb{C})$, we easily obtain the following result (cf. Theorem 3.18).

Corollary 4.2. The inequality $\mathrm{lct}(\mathbb{P}^3, \bar{G}) \geq 1$ holds if the following conditions are satisfied:

- the group G is transitive,
- the group G does not have semi-invariants of degree at most 3,

- the inequality $|\bar{G}| \geq 61$ holds.

The main purpose of this section is to prove the following result (cf. Theorem 1.26).

Theorem 4.3. The inequality $\text{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$ holds if the following conditions are satisfied:

- the group G is primitive,
- the group G does not have semi-invariants of degree at most 4,
- the inequality $|\bar{G}| \geq 169$ holds.

Proof. Suppose that G is primitive and does not have semi-invariants of degree at most 4, the inequality $|\bar{G}| \geq 169$ holds, but $\text{lct}(\mathbb{P}^3, \bar{G}) < 5/4$. Let us derive a contradiction.

There are an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^3 such that

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^3} \sim \mathcal{O}_{\mathbb{P}^3}(4)$$

and a positive rational number $\lambda < 5/4$ such that $(\mathbb{P}^3, \lambda D)$ is strictly log canonical.

Let $S \subset X$ be a minimal center in the set $\text{LCS}(\mathbb{P}^3, \lambda D)$. Note that

$$g(S) \in \text{LCS}(\mathbb{P}^3, \lambda D)$$

for every $g \in \bar{G}$, because the divisor D is \bar{G} -invariant. It follows from Lemma 2.9 that

$$g(S) \cap g'(S) \neq \emptyset \iff g(S) = g'(S)$$

for every $g \in \bar{G} \ni g'$. It follows from Lemma 2.11 that we may assume that

$$\text{LCS}(\mathbb{P}^3, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\},$$

where $\dim(S) \neq 2$, because G has no semi-invariants of degree at most 4.

Let \mathcal{I} be the multiplier ideal sheaf of the log pair $(\mathbb{P}^3, \lambda D)$, and let \mathcal{L} be the log canonical singularities subscheme of the log pair $(\mathbb{P}^3, \lambda D)$. Then there is an exact sequence

$$(4.4) \quad 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{I}) = 0$$

by Theorem 2.6. Then $\dim(S) \neq 0$, because G has no semi-invariants of degree at most 4.

We see that S is a curve. Then it follows from Theorem 2.10 that

- the curve S is a smooth curve of genus g ,
- the inequality $2g - 2 < \deg(S)$ holds.

Let Z be the \bar{G} -orbit of the curve S . Then Z is smooth and $\deg(Z) \leq 6$ by Lemma 2.12. Then

$$2g - 2 < \deg(S) \leq 6,$$

which implies that $g \leq 3$. Note that $Z = \mathcal{L}$ by Remark 2.5, because $(\mathbb{P}^3, \lambda D)$ is log canonical.

The curve Z is not contained in a plane, because G is transitive.

Let r be the number of irreducible components of Z . Then

$$6 \geq \deg(Z) = r \deg(S),$$

which implies that $r \leq 6$. Note that $g = 0$ if $r \geq 3$.

Using the exact sequence 4.4 and the Riemann–Roch theorem, we see that

$$(4.5) \quad 4 = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = r(\deg(S) - g + 1),$$

because $\mathcal{L} = Z$ and $2g - 2 < \deg(S)$. In particular, we see that $r \leq 2$.

One has $\deg(S) \neq 1$, because G is primitive. Thus S is not contained in a plane, because otherwise the \bar{G} -orbit of the plane spanned by S would give a semi-invariant of G of degree 1 or 2.

We have $6 \geq \deg(Z) = r \deg(S) \geq 3r$.

Suppose that $r = 2$. Then $\deg(S) = 3$ and $g = 0$, which contradicts the equalities 4.5.

We see that $Z = S$. Then $g \leq 1$ by Theorem 2.15 and Lemma 2.14, because $|\bar{G}| \geq 169$.

Arguing as in the proof of Theorem 4.1, we see that $g \neq 0$, because G does not have semi-invariants of degree 4. Then $g = 1$ and $\deg(S) = 4$ (see the equalities 4.5). We see that

$$S = Q_1 \cap Q_2,$$

where Q_1 and Q_2 are irreducible quadrics in \mathbb{P}^3 . Let \mathcal{P} be a pencil generated by Q_1 and Q_2 . Put

$$\Lambda = \left\{ \text{Sing}(S_1), \text{Sing}(S_2), \text{Sing}(S_3), \text{Sing}(S_4) \right\} \subset \mathbb{P}^3,$$

where S_1, S_2, S_3, S_4 are singular surfaces in \mathcal{P} . Then $|\Lambda| = 4$ and Λ is \bar{G} -invariant, which is impossible, because G has no semi-invariants of degree 4. \square

Suppose that G is primitive. By [3] and [13], we may assume that $G \subset \text{SL}(4, \mathbb{C})$ and

$$Z(G) \subseteq [G, G],$$

where $Z(G)$ and $[G, G]$ are the center and the commutator of group G , respectively.

Theorem 4.6 (see [3, Chapter VII] or [13, §8.5]). One of the two possibilities holds:

- either G satisfies the hypotheses of Lemma 3.19 or Lemma 3.21,
- or G is one of the following groups:
 - \mathbb{A}_5 or \mathbb{S}_5 ,
 - $\text{SL}(2, \mathbb{F}_5)$,
 - $\text{SL}(2, \mathbb{F}_7)$,
 - $2.\mathbb{A}_6$, which is a central extension of the group $\mathbb{A}_6 \cong \bar{G}$,
 - $2.\mathbb{S}_6$, which is a central extension² of the group $\mathbb{S}_6 \cong \bar{G}$,
 - $2.\mathbb{A}_7$, which is a central extension of the group $\mathbb{A}_7 \cong \bar{G}$,
 - $\tilde{\mathcal{O}}'(5, \mathbb{F}_3)$, which is a central extension of the commutator of the group $\mathcal{O}(5, \mathbb{F}_3)$,
 - in the notation of Example 1.34, a primitive subgroup in N that contains $\alpha(\tilde{\mathbb{H}})$.

Note that if there are two monomorphisms $\iota_1: G \rightarrow \text{SL}(4, \mathbb{C})$ and $\iota_2: G \rightarrow \text{SL}(4, \mathbb{C})$ such that both subgroups $\iota_1(G)$ and $\iota_2(G)$ are primitive, then

- it follows from [3, Chapter VII] that $\iota_1(G)$ and $\iota_2(G)$ are conjugate,
- it may happen that there is no element $g \in \text{SL}(4, \mathbb{C})$ that makes the diagram

$$\begin{array}{ccc} & G & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ \text{SL}(4, \mathbb{C}) & \xrightarrow{\theta_g} & \text{SL}(4, \mathbb{C}) \end{array}$$

commutative, where θ_g is the conjugation by g (cf. [12]).

Lemma 4.7. Suppose that $G \cong 2.\mathbb{A}_6$. Then G has no semi-invariants of degree at most 4.

Proof. Semi-invariants of G are its invariants by Remark 1.25, and G has no odd degree invariants, because G contains a scalar matrix whose non-zero entries are -1 .

To complete the proof, it is enough to prove G has no invariants of degree 4.

Let $V \cong \mathbb{C}^4$ be the irreducible representation of the group G that corresponds to the embedding $G \subset \text{SL}(4, \mathbb{C})$. Without loss of generality, we may assume that

$$\Lambda^2 V \cong \mathbb{C}^6$$

is a permutation representation of the group $G/Z(G) \cong \mathbb{A}_6$, because G has two four-dimensional irreducible representations, which give one subgroup $G \subset \text{SL}(4, \mathbb{C})$ up to conjugation.

²There are three non-isomorphic non-trivial central extensions of the group \mathbb{S}_6 with the center \mathbb{Z}_2 , two of which are embedded in $\text{SL}(4, \mathbb{C})$ (cf. [12]). But up to conjugation there is one subgroup of $\text{PGL}(4, \mathbb{C})$ isomorphic to \mathbb{S}_6 .

Let χ be its character, and let χ_4 be the character of the representation $\text{Sym}^4(V)$. Then

$$\chi_4(g) = \frac{1}{24} \left(\chi(g)^4 + 6\chi(g)^2\chi(g^2) + 3\chi(g^2)^2 + 8\chi(g)\chi(g^3) + 6\chi(g^4) \right)$$

for every $g \in G$. The values of the characters χ and χ_4 are listed in the following table.

	$[5, 1]_{10}$	$[5, 1]_5$	$[4, 2]_8$	$[3, 3]_6$	$[3, 3]_3$	$[3, 1, 1, 1]_6$	$[3, 1, 1, 1]_3$	$[2, 2, 1, 1]_4$	z	e
#	144	144	180	40	40	40	40	90	1	1
χ	1	-1	0	-1	1	2	-2	0	-4	4
χ_4	0	0	-1	2	2	-4	-4	3	35	35

where the first row lists the types of the elements in G (for example, the symbol $[5, 1]_{10}$ denotes the set³ of order 10 elements whose image in \mathbb{A}_6 is a product of disjoint cycles of length 5 and 1), and z and e are the non-trivial element in the center of G and the identity element, respectively.

Now one can check that the inner product of the character χ_4 and the trivial character is zero, which implies that the subgroup G does not have invariants of degree 4. \square

Corollary 4.8. If $G \cong 2.\mathbb{S}_6$ or $G \cong 2.\mathbb{A}_7$, then G has no semi-invariants of degree at most 4.

Proof. Recall that these groups contain $2.\mathbb{A}_6$ and we can apply Lemma 4.7. \square

Recall that we assume that G is primitive and $Z(G) \subseteq [G, G]$.

Lemma 4.9. The following two conditions are equivalent:

- the subgroup G has no semi-invariants of degree at most 4,
- the subgroup G is one of the following groups:
 - $2.\mathbb{A}_6$, $2.\mathbb{S}_6$ or $2.\mathbb{A}_7$,
 - $\tilde{\mathcal{O}}'(5, \mathbb{F}_3)$,
 - one of the four groups constructed in Example 1.34.

Proof. Let d be the smallest positive number such G has an semi-invariant of degree d . Then

- if $G \cong 2.\mathbb{A}_6$, then the inequality $d \geq 5$ holds by Lemma 4.7,
- if $G \cong 2.\mathbb{S}_6$ or $G \cong 2.\mathbb{A}_7$, then the inequality $d \geq 5$ holds⁴ by Corollary 4.8,
- if $G \cong \text{SL}(2, \mathbb{F}_7)$, then the equality $d = 4$ holds by [27] and Remark 1.25,
- if $G \cong \tilde{\mathcal{O}}'(5, \mathbb{F}_3)$, then the equality $d = 12$ holds by [29] and Remark 1.25.

Suppose that $G \cong \text{SL}(2, \mathbb{F}_5) \cong 2.\mathbb{A}_5$. Then there is a \bar{G} -invariant smooth rational cubic curve

$$C \subset \mathbb{P}^3,$$

because the representation $G \rightarrow \text{GL}(4, \mathbb{C})$ is a symmetric square of a two-dimensional representation of the group G . By Theorem 4.1, the inequality $d \leq 4$ holds⁵.

Let us use the notation of Example 1.34. By Theorem 4.6 and by Lemmas 2.13, 3.19 and 3.21, to complete the proof we may assume that G is a primitive subgroup in N that contains $\alpha(\tilde{\mathbb{H}})$.

It follows from [34, Lemma 3.18] that the group $\tilde{\mathbb{H}}$ has no invariants of degree less than 4, and its invariants of degree 4 form a five-dimensional vector space W .

The group $\beta(G)$ naturally acts on W . The following are equivalent:

- the subgroup G has an invariant of degree 4,
- the representation W has a one-dimensional subrepresentation of the group $\beta(G)$.

³ Note that these sets do not coincide with conjugacy classes. For example, the image of the set of the elements of type $[5, 1]_{10}$ under the natural projection $2.\mathbb{A}_6 \rightarrow \mathbb{A}_6$ is a union of two different conjugacy classes in \mathbb{A}_6 .

⁴One can check by direct computation that $d = 8$ if $G \cong 2.\mathbb{A}_6$ or $G \cong 2.\mathbb{S}_6$ or $G \cong 2.\mathbb{A}_7$.

⁵One can show that $d = 4$ in this case.

It follows from [31] that if $G = N$, then W is an irreducible representation of $\beta(G) = \mathbb{S}_6$.

It follows from [3, §123] that, up to conjugation, there exist exactly 9 possibilities for the subgroup $G \subset N$ such that G is primitive. These possibilities are listed in the following table:

Label of the group G	$\beta(G)$	Generators of the subgroup $\beta(G) \subseteq \mathbb{S}_6$	Splitting type
13°	\mathbb{Z}_5	(24635)	1, 1, 1, 1, 1
14°	$\mathbb{Z}_5 \rtimes \mathbb{Z}_2$	(24635), (36)(45)	1, 2, 2
15°	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	(24635), (3465)	1, 2, 2
16°	\mathbb{A}_5	(24635), (34)(56)	1, 4
17°	\mathbb{A}_5	(24635), (12)(36)	5
18°	\mathbb{S}_5	(24635), (56)	1, 4
19°	\mathbb{S}_5	(24635), (12)(34)(56)	5
20°	\mathbb{A}_6	(24635), (12)(34)	5
21°	\mathbb{S}_6	(24635), (12)	5

where the first column lists the labels of the subgroup G according to [3, §123] and the last column lists the dimensions of the irreducible $\beta(G)$ -subrepresentation of in W .

Note that $\tilde{\mathbb{H}}$ has no semi-invariants of degree 3, because $\tilde{\mathbb{H}}$ has no invariants of degree 3, the center of the group $\tilde{\mathbb{H}}$ coincides with its commutator and acts non-trivially on cubic forms.

The groups constructed in Example 1.34 are the subgroups 21° , 20° , 19° , 17° . We see that

- the equality $d \leq 4$ holds if G is the subgroup 13° , 14° , 15° , 16° or 18° ,
- if G is the subgroup 17° , 19° , 20° or 21° , then the subgroup G has neither semi-invariants of degree less than 4, nor invariants of degree 4.

Note that to prove the inequality $d \geq 5$ in the case when G is the subgroup 17° , 19° , 20° or 21° , it is enough to prove that the subgroup 17° does not have semi-invariants of degree 4.

Suppose that G is the subgroup 17° , and suppose, in addition, that G does have a semi-invariant Φ of degree 4. Let us show that this assumption leads to a contradiction⁶.

Note that Φ is not $\tilde{\mathbb{H}}$ -invariant, because Φ is not G -invariant and

$$G/\tilde{\mathbb{H}} \cong \beta(G) \cong \mathbb{A}_5$$

is a simple group. Let Z be the center of the group $\tilde{\mathbb{H}}$. Put $\bar{\mathbb{H}} = \phi(\tilde{\mathbb{H}})$. Then

$$\tilde{\mathbb{H}}/Z \cong \bar{\mathbb{H}} \cong \mathbb{Z}_2^4,$$

and Z acts trivially on Φ . Thus, there is a homomorphism $\xi: \bar{\mathbb{H}} \rightarrow \mathbb{C}^*$ such that

$$\ker(\xi) \neq \bar{\mathbb{H}},$$

which implies that $\ker(\xi) \cong \mathbb{Z}_2^3$, because $\text{im}(\chi)$ is a cyclic group,

Let $\theta: \bar{G} \rightarrow \text{Aut}(\bar{\mathbb{H}})$ be the homomorphism such that

$$\theta(g)(h) = ghg^{-1} \in \bar{\mathbb{H}} \cong \mathbb{Z}_2^4$$

for all $g \in \bar{G}$ and $h \in \bar{\mathbb{H}}$. Consider $\bar{\mathbb{H}}$ as a vector space over \mathbb{F}_2 . Then θ induces a monomorphism

$$\tau: \beta(G) \rightarrow \text{GL}(4, \mathbb{F}_2),$$

and $\ker(\xi)$ is a $\text{im}(\tau)$ -invariant subspace. But $\text{im}(\tau) \cong \mathbb{A}_5$ has no non-trivial three-dimensional representations over \mathbb{F}_2 , because $|\text{GL}(3, \mathbb{F}_2)| = 168$ is not divisible by $|\mathbb{A}_5| = 60$. Thus, we see that there is a non-zero element $t \in \bar{\mathbb{H}}$ such that t is $\text{im}(\tau)$ -invariant.

⁶ One can check by direct computation that $d = 8$ if G is the subgroup 17° , 19° , 20° or 21° .

Let $F \subset \mathrm{GL}(4, \mathbb{F}_2)$ be the stabilizer of t . Then

$$\mathbb{A}_5 \cong \mathrm{im}(\tau) \subset F,$$

which is impossible, because $|F| = 1344$ is not divisible by $|\mathbb{A}_5| = 60$. \square

5. FIVE-DIMENSIONAL CASE

Let \bar{G} be a finite primitive subgroup in $\mathrm{Aut}(\mathbb{P}^4)$. There is a subgroup $G \subset \mathrm{SL}(5, \mathbb{C})$ such that

$$\phi(G) = \bar{G} \subset \mathrm{Aut}(\mathbb{P}^4) \cong \mathrm{PGL}(5, \mathbb{C})$$

and $Z(G) \subseteq [G, G]$ (see [5] and [13]), where $\phi: \mathrm{GL}(5, \mathbb{C}) \rightarrow \mathrm{Aut}(\mathbb{P}^4)$ is the natural projection.

Theorem 5.1 (see [5] or [13, §8.5]). The subgroup G is one of the following groups:

- $\mathbb{A}_5, \mathbb{A}_6, \mathbb{S}_5$ or \mathbb{S}_6 ,
- $\mathrm{PSL}(2, \mathbb{F}_{11})$,
- $\mathbb{O}'(3, \mathbb{F}_5)$, which is the commutator of the group $\mathbb{O}(3, \mathbb{F}_5)$,
- in the notation of Example 1.35, a primitive subgroup of HM that contains $\alpha(\mathbb{H})$.

Note that if there are two monomorphisms $\iota_1: G \rightarrow \mathrm{SL}(5, \mathbb{C})$ and $\iota_2: G \rightarrow \mathrm{SL}(5, \mathbb{C})$ such that both subgroups $\iota_1(G)$ and $\iota_2(G)$ are primitive, then $\iota_1(G)$ and $\iota_2(G)$ are conjugate.

Lemma 5.2. Suppose that G is one of the groups $\mathbb{A}_5, \mathbb{A}_6, \mathbb{S}_5, \mathbb{S}_6, \mathrm{PSL}(2, \mathbb{F}_{11})$ or $\mathbb{O}'(3, \mathbb{F}_5)$. Then

- the subgroup G has an invariant of degree at most 4,
- the inequality $\mathrm{lct}(\mathbb{P}^4, \bar{G}) \leq 4/5$ holds.

Proof. If G is $\mathbb{A}_5, \mathbb{A}_6, \mathbb{S}_5$ or \mathbb{S}_6 , then G has an invariant of degree 2 by Lemma 2.13.

If $G \cong \mathbb{O}'(3, \mathbb{F}_5)$, then G has an invariant of degree 4 (see [7]).

If $G \cong \mathrm{PSL}(2, \mathbb{F}_{11})$, then G has an invariant of degree 3 (see [1]). \square

Let us use the notation of Example 1.35. Suppose that $\alpha(\mathbb{H}) \subsetneq G \subseteq \mathrm{HM}$.

Lemma 5.3. The following are equivalent:

- the subgroup G has no semi-invariants of degree at most 5,
- either $G = \mathrm{HM}$ or G is a subgroup of HM of index 5.

Proof. Let V be the vector space of \mathbb{H} -invariant forms of degree 5. Then the group

$$\mathrm{HM}/\mathbb{H} \cong \mathrm{SL}(2, \mathbb{F}_5) \cong 2.\mathbb{A}_5$$

naturally acts on the vector space V . Moreover, by [16, Theorem 3.5]

$$V = V' \oplus V'',$$

where V' and V'' are three-dimensional $\mathrm{im}(\beta)$ -invariant linear subspaces that arise from two non-equivalent three-dimensional representations of the group \mathbb{A}_5 , respectively.

Therefore, we see that the following are equivalent:

- the group G has a semi-invariant of degree 5,
- the subspace V' has a $\beta(G)$ -invariant one-dimensional subspace.

Let $Z \cong \mathbb{Z}_2$ be the center of the group $\mathrm{HM}/\mathbb{H} \cong 2.\mathbb{A}_5$. Then $2.\mathbb{A}_5/Z \cong \mathbb{A}_5$ and

- either $\beta(G)$ is cyclic,
- or $Z \subseteq \beta(G)$ and $\beta(G)/Z$ is one of the following subgroups of \mathbb{A}_5 :
 - dihedral group of order 6,
 - dihedral group of order 10,
 - the group $\mathbb{Z}_2 \times \mathbb{Z}_2$,
 - the group \mathbb{A}_4 ,
 - the group \mathbb{A}_5 .

If $\beta(G)$ is cyclic, then V' is a sum of one-dimensional $\beta(G)$ -invariant linear subspaces. Hence we may assume that $Z \subseteq \beta(G)$. Recall that $Z \cong \mathbb{Z}_2$ acts trivially on V' . Thus, if

$$\beta(G)/Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

then V' is a sum of one-dimensional $\beta(G)$ -invariant subspaces.

If $\beta(G)/Z$ is a dihedral group, then V' must have one-dimensional $\beta(G)$ -invariant subspace, because irreducible representations of dihedral groups are one-dimensional or two-dimensional.

If $\beta(G)/Z \cong \mathbb{A}_5$ or $\beta(G)/Z \cong \mathbb{A}_4$, then V' is an irreducible representation of $\beta(G)/Z$, which implies that V' is an irreducible representation of the group $\beta(G)$.

Now using Corollary A.2, we complete the proof. \square

Suppose that G is a subgroup of the group $\mathbb{H}\mathbb{M}$ of index 5.

Lemma 5.4. Let Λ be a \bar{G} -invariant finite subset of \mathbb{P}^4 . Then $|\Lambda| \geq 10$.

Proof. The inequality follows from Lemma 5.3 and Corollary A.2. \square

The main purpose of this section is to prove the following result.

Theorem 5.5. The inequality $\text{lct}(\mathbb{P}^4, \bar{G}) \geq 6/5$ holds.

Corollary 5.6. The inequality $\text{lct}(\mathbb{P}^4, \bar{\mathbb{H}\mathbb{M}}) \geq 6/5$ holds, where $\bar{\mathbb{H}\mathbb{M}} = \phi(\mathbb{H}\mathbb{M})$.

Let us prove Theorem 5.5. Suppose that $\text{lct}(\mathbb{P}^4, \bar{G}) < 6/5$. Then there are a rational positive number $\lambda < 6/5$ and an effective \bar{G} -invariant \mathbb{Q} -divisor D on \mathbb{P}^5 such that

$$D \sim_{\mathbb{Q}} -K_{\mathbb{P}^4} \sim \mathcal{O}_{\mathbb{P}^4}(5),$$

and the log pair $(\mathbb{P}^4, \lambda D)$ is strictly log canonical.

Corollary 5.7. The locus $\text{LCS}(\mathbb{P}^4, \lambda D)$ is \bar{G} -invariant.

Let S be a minimal center in $\text{LCS}(\mathbb{P}^4, \lambda D)$, let Z be the \bar{G} -orbit of the subvariety $S \subset \mathbb{P}^4$, and let r be the number of irreducible components of the subvariety Z . We may assume that

$$\text{LCS}(\mathbb{P}^4, \lambda D) = \bigcup_{g \in \bar{G}} \{g(S)\}$$

by Lemma 2.11. Then $\text{Supp}(Z) = \text{LCS}(\mathbb{P}^4, \lambda D)$. It follows from Lemma 2.9 that

$$g(S) \cap g'(S) \neq \emptyset \iff g(S) = g'(S)$$

for every $g \in \bar{G} \ni g'$. Then $\deg(Z) = r \deg(S)$.

Lemma 5.8. The equality $\dim(S) = 3$ is impossible.

Proof. Suppose that $\dim(S) = 3$. Then there is $\mu \in \mathbb{Q}$ such that $\mu \geq 1/\lambda$ and

$$D = \mu Z + \Omega \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^4}(5),$$

where Ω is an effective \bar{G} -invariant \mathbb{Q} -divisor on \mathbb{P}^4 . Then

$$\deg(Z) \leq \frac{5}{\mu} \leq \frac{5}{\lambda} < 6,$$

which is impossible, because G has no semi-invariants of degree at most 5 by Lemma 5.3. \square

Let \mathcal{I} be the multiplier ideal sheaf of the log pair $(\mathbb{P}^4, \lambda D)$, and let \mathcal{L} be the log canonical singularities subscheme of the log pair $(\mathbb{P}^4, \lambda D)$. By Theorem 2.6, there is an exact sequence

$$(5.9) \quad 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \longrightarrow H^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^4}(n)) \longrightarrow 0$$

for every $n \geq 1$. Note that $Z = \mathcal{L}$ by Remark 2.5.

Lemma 5.10. The equality $\dim(S) = 0$ is impossible.

Proof. Suppose that $\dim(S) = 0$. Then it follows from the exact sequence 5.9 that the inequality

$$\left| \text{LCS}(\mathbb{P}^4, \lambda D) \right| \leq h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = 5,$$

holds. But $\text{LCS}(\mathbb{P}^4, \lambda D)$ is \bar{G} -invariant, which is impossible by Lemma 5.4. \square

Lemma 5.11. The equality $\dim(S) = 1$ is impossible.

Proof. Suppose that $\dim(S) = 1$. Then it follows from Theorem 2.10 that

- the curve S is a smooth curve of genus g ,
- the inequality $2g - 2 < \deg(S)$ holds.

It follows from Lemma 2.12 that $\deg(Z) \leq 10$. Then

$$2g - 2 < \deg(S) \leq 10,$$

which implies that $g \leq 5$. The curve Z is not contained in a hyperplane, because G is transitive. Then $10 \geq \deg(Z) = r \deg(S)$, which implies that $r \leq 10$.

Using the exact sequence 5.9 and the Riemann–Roch theorem, we see that

$$(5.12) \quad 5 = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = r(\deg(S) - g + 1),$$

because $\mathcal{L} = Z$ and $2g - 2 < \deg(S)$. Thus, either $r = 1$ or $r = 5$.

Suppose that $r = 5$. Then $\deg(S) = 2$ and $g = 0$, which contradicts the equalities 5.12.

We see that $r = 1$. Then $g = \deg(S) - 4$. Using the exact sequence 5.9, we see that

$$15 = h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(2)) = 2\deg(S) - g + 1 = \deg(S) + 5,$$

which gives $g = 6$, which is impossible by Theorem 2.15 and Lemma 2.14, because $|\bar{G}| \geq 421$. \square

We see that $\dim(S) = 2$. Then $\deg(Z) \leq 10$ by Lemma 2.12. It follows from Theorem 2.10 that

- the surface S is normal and has at most rational singularities,
- there are an effective \mathbb{Q} -divisor B_S and an ample \mathbb{Q} -divisor Δ on the surface S such that

$$K_S + B_S + \Delta \equiv \mathcal{O}_{\mathbb{P}^4}(1)|_S,$$

and the log pair (S, B_S) has Kawamata log terminal singularities.

Corollary 5.13. The equality $r = 1$ holds.

Proof. Two irreducible surfaces in \mathbb{P}^4 have non-empty intersection. \square

Thus, we see that the surface $S = Z$ is \bar{G} -invariant.

Lemma 5.14. The surface S is not contained in a hyperplane in \mathbb{P}^4 .

Proof. The required assertion follows from the fact that G is transitive. \square

Lemma 5.15. The surface S is not contained in a quadric hypersurface in \mathbb{P}^4 .

Proof. Suppose that there is a quadric hypersurface $Q \subset \mathbb{P}^4$ such that $S \subset Q$. Then

- the quadric Q is irreducible by Lemma 5.14,
- it follows from Lemma 5.3 that there is a quadric hypersurface $Q' \subset \mathbb{P}^4$ such that

$$S \subseteq Q \cap Q',$$

because otherwise the quadric Q would be \bar{G} -invariant,

- the quadric Q' is irreducible by Lemma 5.14.

Suppose that $S = Q \cap Q'$. Then S is singular by Lemma 2.14 and [17]. It follows from [15] that

$$|\text{Sing}(S)| \leq 4,$$

because S has canonical singularities since S is a complete intersection that has Kawamata log terminal singularities. But $\text{Sing}(S)$ is \bar{G} -invariant, which contradicts Lemma 5.4.

We see that $S \neq Q \cap Q'$. Therefore, it follows from Lemma 5.14 that

- either S is a cone over a smooth rational cubic curve,
- or S is a smooth cubic scroll.

Suppose that S is a cone. Then its vertex is \bar{G} -invariant, which contradicts Lemma 5.4.

We see that S is a smooth cubic scroll. Then there is a unique line $L \subset S$ such that $L^2 = -1$, which implies that L must be \bar{G} -invariant, which is impossible, because G is primitive. \square

Let H be a hyperplane section of the surface $S \subset \mathbb{P}^4$.

Lemma 5.16. The equalities $H \cdot H = -H \cdot K_S = 5$ and $\chi(\mathcal{O}_S) = 0$ hold.

Proof. It follows from Corollary A.2 that there is $m \geq 0$ such that the equality

$$h^0(\mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}) = 5m$$

holds. Let us show that this is possible only if $H \cdot H = -H \cdot K_S = 5$ and $\chi(\mathcal{O}_S) = 0$.

It follows from the Riemann–Roch theorem and Theorem 2.6 that

$$(5.17) \quad h^0(\mathcal{O}_S(nH)) = \chi(\mathcal{O}_S(nH)) = \chi(\mathcal{O}_S) + \frac{n^2}{2}(H \cdot H) - \frac{n}{2}(H \cdot K_S)$$

for any $n \geq 1$. It follows from Lemma 5.14, the equalities 5.17 and the exact sequence 5.9 that

$$(5.18) \quad 5 = h^0(\mathcal{O}_S(H)) = \chi(\mathcal{O}_S) + \frac{1}{2}(H \cdot H) - \frac{1}{2}(H \cdot K_S),$$

and it follows from Lemma 5.15, the equalities 5.17 and the exact sequence 5.9 that

$$(5.19) \quad 10 = h^0(\mathcal{O}_S(2H)) = \chi(\mathcal{O}_S) + 2(H \cdot H) - (H \cdot K_S).$$

It follows from Lemmas 2.12, 5.14 and 5.15 that $4 \leq H \cdot H = \deg(S) \leq 10$.

Suppose that $H \cdot H = 10$. It follows from the equalities 5.18 and 5.19 that $\chi(\mathcal{O}_S) = 5$ and

$$H \cdot K_S = H \cdot H = 10,$$

which is impossible, because $H \equiv K_S + B_S + \Delta$, where Δ is ample and B_S is effective.

It follows from the equalities 5.18 and 5.19 that

$$H \cdot K_S = 3\chi(\mathcal{O}_S) - 5 = 3(H \cdot H) - 20.$$

It follows from the equalities 5.17 and the exact sequence 5.9 that

$$h^0(\mathcal{O}_{\mathbb{P}^4}(3) \otimes \mathcal{I}) = 35 - h^0(\mathcal{O}_S(3H)) = 35 - \left(\chi(\mathcal{O}_S) + \frac{9}{2}(H \cdot H) - \frac{3}{2}(H \cdot K_S) \right) = 5m,$$

which implies that $H \cdot H = 5$, $\chi(\mathcal{O}_S) = 0$ and $H \cdot K_S = -5$, because $4 \leq H \cdot H \leq 9$. \square

Let $\pi: U \rightarrow S$ be the minimal resolution of the surface S . Then $\kappa(U) = -\infty$ and

$$1 - h^1(\mathcal{O}_U) = 1 - h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = h^2(\mathcal{O}_U) = h^0(\mathcal{O}_U(K_U)) = 0,$$

because S has rational singularities and $\kappa(U) = -\infty$ since $H \cdot K_S = -5 < 0$.

Corollary 5.20. The surface S is birational to $E \times \mathbb{P}^1$, where E is smooth elliptic curve.

By Lemma 2.14, there is a monorphism $\xi: \bar{G} \rightarrow \text{Aut}(Y)$, which contradicts Corollary A.11.

The obtained contradiction completes the proof of Theorem 5.5.

APPENDIX A. HORROCKS–MUMFORD GROUP

Let \mathbb{H} be the Heisenberg group of all unipotent 3×3 -matrices with entries in \mathbb{F}_5 . Then

$$\mathbb{H} = \left\langle x, y, z \mid x^5 = y^5 = z^5 = 1, \quad xz = zx, \quad yz = zy, \quad xy = zyx \right\rangle$$

for some $x, y, z \in \mathbb{H}$. There is a monomorphism $\rho: \mathbb{H} \rightarrow \mathrm{SL}(5, \mathbb{C})$ such that

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \zeta & 0 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 & 0 \\ 0 & 0 & 0 & \zeta^4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where ζ is a non-trivial fifth root of unity. Let us identify \mathbb{H} with $\mathrm{im}(\rho)$. Then $Z(\mathbb{H}) \cong \mathbb{Z}_5$ and

$$\begin{pmatrix} \zeta & 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & \zeta \end{pmatrix} \in Z(\mathbb{H}),$$

where $Z(\mathbb{H})$ is the center of \mathbb{H} . Let $\phi: \mathrm{GL}(5, \mathbb{C}) \rightarrow \mathrm{PGL}(5, \mathbb{C})$ be the natural projection.

Lemma A.1 ([16, §1]). Let $\chi: \mathbb{H} \rightarrow \mathrm{GL}(N, \mathbb{C})$ be an irreducible representation of \mathbb{H} . Then

- either $N = 1$ and $Z(\mathbb{H}) \subseteq \ker(\chi)$,
- or N is divisible by 5.

Take $n \in \mathbb{Z}_{\geq 0}$. Then \mathbb{H} naturally acts on $H^0(\mathcal{O}_{\mathbb{P}^4}(n))$.

Corollary A.2. Let V be a \mathbb{H} -invariant subspace in $H^0(\mathcal{O}_{\mathbb{P}^4}(n))$. Then

- either $\dim(V)$ is divisible by 5,
- or n is divisible by 5.

Let $\mathbb{H}\mathbb{M} \subset \mathrm{SL}(5, \mathbb{C})$ be the normalizer of the subgroup \mathbb{H} . Then there is an exact sequence

$$1 \longrightarrow \mathbb{H} \xrightarrow{\alpha} \mathbb{H}\mathbb{M} \xrightarrow{\beta} \mathrm{SL}(2, \mathbb{F}_5) \longrightarrow 1,$$

and it follows from [16, §1] that there is a subgroup $\mathbb{M} \subset \mathbb{H}\mathbb{M}$ such that

$$\mathbb{H}\mathbb{M} = \mathbb{H} \rtimes \mathbb{M}$$

and $\mathbb{M} \cong \beta(\mathbb{M}) = \mathrm{SL}(2, \mathbb{F}_5) \cong 2.A_5$. Put $\bar{\mathbb{H}} = \phi(\mathbb{H})$ and $\overline{\mathbb{H}\mathbb{M}} = \phi(\mathbb{H}\mathbb{M})$. Then

$$\overline{\mathbb{H}\mathbb{M}}/\bar{\mathbb{H}} \cong \mathrm{SL}(2, \mathbb{F}_5)$$

and $\bar{\mathbb{H}} \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Let $Z(\mathbb{H}\mathbb{M})$ be the center of the group $\mathbb{H}\mathbb{M}$. Then $Z(\mathbb{H}\mathbb{M}) = Z(\mathbb{H}) \cong \mathbb{Z}_5$.

Corollary A.3. The group $\overline{\mathbb{H}\mathbb{M}}$ is isomorphic to $\mathbb{H}\mathbb{M}/Z(\mathbb{H}\mathbb{M})$.

Let G be a subgroup of the group $\mathbb{H}\mathbb{M}$ of index 5. Then

$$G \cong \mathbb{H} \rtimes 2.A_4 \subset \mathbb{H} \rtimes 2.A_5,$$

and $|\bar{G}| = 600$, where $\bar{G} = \phi(G)$. Let $Z(G)$ be the center of the group G . Then

$$Z(G) = Z(\mathbb{H}\mathbb{M}) = Z(\mathbb{H}) \cong \mathbb{Z}_5.$$

Lemma A.4. Let g be an element of the group \bar{G} such that

$$gh = hg \in \bar{G}$$

for every element $h \in \bar{\mathbb{H}}$. Then $g \in \bar{\mathbb{H}}$.

Proof. The required assertion follows from [16, §1]. \square

Lemma A.5. Let F be a proper normal subgroup of $2.A_4$. Then

- either $F \cong \mathbb{Z}_2$ is a center of the group $2.A_4$,
- or $F \cong \mathbb{Q}_8$, where \mathbb{Q}_8 is the quaternion group of order 8.

Proof. The only nontrivial normal subgroup of the group A_4 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. \square

Lemma A.6. The group $\bar{\mathbb{H}}$ contains no proper non-trivial subgroups that are normal in \bar{G} .

Proof. Let $\theta: \bar{\mathbb{H}\mathbb{M}} \rightarrow \text{Aut}(\bar{\mathbb{H}})$ be the homomorphism such that

$$\theta(g)(h) = ghg^{-1} \in \bar{\mathbb{H}}$$

for all $g \in \bar{\mathbb{H}\mathbb{M}}$ and $h \in \bar{\mathbb{H}}$. Then $\ker(\theta) = \bar{\mathbb{H}}$ by Lemma A.4.

The homomorphism θ induces an isomorphism $\tau: \mathbb{M} \rightarrow \text{SL}(2, \mathbb{F}_5)$.

Let $F \subset \mathbb{M}$ be a subgroup such that $\beta(F) = \beta(G) \cong 2.A_4$. Then $G = \mathbb{H} \rtimes F$.

Suppose that the group $\bar{\mathbb{H}}$ contains a proper non-trivial subgroup that is a normal subgroup of the group \bar{G} . Let us consider $\bar{\mathbb{H}}$ as a two-dimensional vector space over \mathbb{F}_5 . Then

$$\mathbb{F}_5^2 \cong \bar{\mathbb{H}} = V_0 \oplus V_1,$$

where V_0 and V_1 are one-dimensional $\tau(F)$ -invariant subspaces, since $|2.A_4| = 24$ is coprime to 5.

By Lemma A.4, the homomorphism τ induces a monomorphism

$$F \longrightarrow \text{GL}(1, \mathbb{F}_5) \times \text{GL}(1, \mathbb{F}_5) \cong \mathbb{Z}_4 \times \mathbb{Z}_4,$$

which implies that F is an abelian group, which is not the case. \square

Lemma A.7. The group \bar{G} does not contain proper normal subgroups not containing $\bar{\mathbb{H}}$.

Proof. Suppose that \bar{G} contains a normal subgroup \bar{G}' such that $\bar{\mathbb{H}} \not\subseteq \bar{G}'$. Then

$$\bar{G}' \cap \bar{\mathbb{H}} = \{e\},$$

by Lemma A.6, where e is the identity element in \bar{G} . Hence

$$\bar{G}' \cong \beta(\bar{G}') \subseteq \beta(\bar{G}) \cong 2.A_4,$$

which implies that \bar{G}' is isomorphic to a normal subgroup of the group $2.A_4$.

Let \bar{Z} be the center of \bar{G}' . Then \bar{Z} is a normal subgroup of the group \bar{G} . Thus, we have

$$\bar{Z} \cong \mathbb{Z}_2$$

by Lemma A.5. Hence \bar{Z} is contained in the center of \bar{G} , which contradicts Lemma A.4. \square

Lemma A.8. Let E be a smooth elliptic curve. Then there is no monomorphism $\bar{G} \rightarrow \text{Aut}(E)$.

Proof. Let us consider E as an abelian group. Then there is an exact sequence

$$1 \longrightarrow E \xrightarrow{\iota} \text{Aut}(E) \xrightarrow{v} \mathbb{Z}_n \longrightarrow 1$$

for some $n \in \{2, 4, 6\}$.

Suppose that there is a monomorphism $\theta: \bar{G} \rightarrow \text{Aut}(E)$. Then

$$\theta(\bar{\mathbb{H}}) \subset \iota(E),$$

because $\iota(E)$ contains all the elements of $\text{Aut}(E)$ of order 5.

Let g be any element of \bar{G} such that $\theta(g) \in \iota(E)$. Then

$$\theta(g)\theta(h) = \theta(h)\theta(g)$$

for every $h \in \bar{\mathbb{H}}$, because $\iota(E)$ is an abelian group, and thus $g \in \bar{\mathbb{H}}$ by Lemma A.4. Hence

$$\theta(\bar{G}) \cap \iota(E) = \theta(\bar{\mathbb{H}}),$$

which implies that $v(\bar{G}) \cong \beta(\bar{G}) \cong 2.A_4$, which is absurd. \square

The main purpose of this section is to prove the following result.

Theorem A.9. Let E be a smooth elliptic curve. Then there is no exact sequence of groups

$$(A.10) \quad 1 \longrightarrow G' \xrightarrow{\iota} \bar{G} \xrightarrow{v} G'' \longrightarrow 1,$$

where G' and G'' are subgroups of the groups $\text{Aut}(\mathbb{P}^1)$ and $\text{Aut}(E)$, respectively.

Proof. Suppose that the exact sequence of groups A.10 does exist. Then ι is not an isomorphism, because the group $\text{Aut}(\mathbb{P}^1)$ does not contain subgroups isomorphic to \bar{G} .

The monomorphism v is not an isomorphism by Lemma A.8. Then

$$\bar{\mathbb{H}} \subset \iota(G')$$

by Lemma A.7. But $\text{Aut}(\mathbb{P}^1)$ contain no subgroups isomorphic to $\bar{\mathbb{H}}$, which is a contradiction. \square

Corollary A.11. There is no monomorphism $\bar{G} \rightarrow \text{Bir}(E \times \mathbb{P}^1)$, where E is smooth elliptic curve.

We believe that there is a simpler proof of Theorem A.9.

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